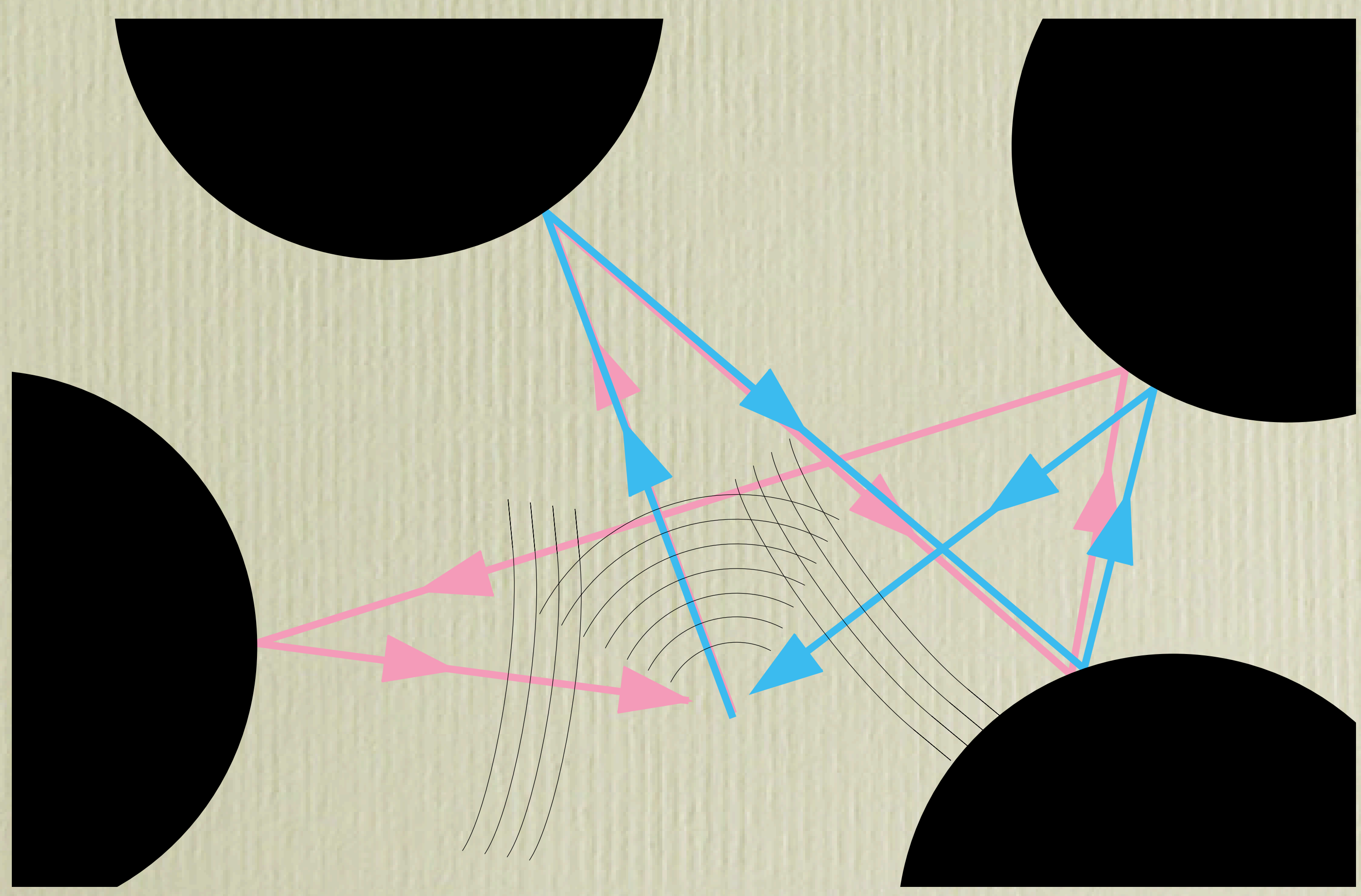
Real Quantum Chaos

- thanks to Lev Kaplan, Tulane
- NSF, LANL

Quantum chaos: the study of the quantum mechanics of classically chaotic systems

Classical chaos: major hallmark is exponential sensitivity to initial conditions (positive Lyopunov exponents)



“Real” quantum chaos means we deal with finite h, not only asymptotics

$$G(q,q',t) \;=\; \frac{1}{\sqrt{2\pi i\hbar}}\sum_k \left(\frac{\partial^2 S_k(q,q',t)}{\partial q\partial q'}\right)^{1/2} e^{\frac{i}{\hbar}(S_k(q,q',t)+i\nu_k)}$$

$S_k(q,q',t)$ is the classical action for the k^{th} *classical* path

Quantum Chaos

Random Matrix Theory

Corrections to RMT

Random Matrix Theory

Wigner, 1950,59   Dyson, 1962,63 .....

Random matrix ensemble, probability of H is

$$P(H) = e^{-\beta \; Tr[H^2]}$$

Energy level spacing distribution, GOE:

$$P(s) \sim s \; e^{-\pi s^2/4}$$

Quantum Chaos and RMT

- Bohigas, Gianonni, and Schmit (1984): conjecture that classically chaotic systems give rise to random matrix statistics when quantized
- Berry (1983): Semiclassical arguments for random superposition of plane waves in classically chaotic systems
- Andreev, Agam, Simons, Altschuler (1996): "We prove the BGS conjecture..."
- Dozens of papers deriving every imaginable observable from RMT
- Semiclassical theories yielding random matrix statistics

QC = RMT?

No, but RMT *is* a foundation on which we can build better theories.

RMT is practically void of dynamics: infinite Lyopunov exponent

Random matrix ensembles need additional constraints necessitated by dynamics.

What is actually proven?

Schnirlman, Zelditch, Colin de Verdiere:

$$\langle\psi_n|A|\psi_n\rangle = Tr[\delta(E - H(p,q))A(p,q)]$$

except for a set of ψ_n vanishing measure as $h \to 0$.

The Weyl symbol $A(p,q)$ is a "macroscopic" operator, independent of $\hbar$.

Does not imply Gaussian random eigenfunctions

Maximum entropy method

Maximum entropy method (unbiased except for normalization):

$$\mathcal{F} = \int dH_{11}\ dH_{12} \ldots \left(P(H) \ln P(H) + \beta P(H) Tr[H^2]\right)$$

$$P(H) \sim e^{-\beta Tr[H^2]}$$

Maximum entropy method (unbiased except for normalization and dynamical constraints):

$$\mathcal{F} = \int dH_{11}\ dH_{12} \ldots \left(P(H) \ln P(H) + \beta P(H) Tr[H^2] + \int \mu(t) P(H) f(H,t)\ dt\right)$$

e.g., $f(H,t) = \langle 1 | e^{-iHt/\hbar} | 1 \rangle$.

$$P(H) \sim e^{-\beta Tr[H^2] - \int \mu(t) f(H,t)\ dt}$$

-destroys invariance under arbitrary unitary transformation.

-good! The first lesson of dynamics: all states are not born equal.

- implies "local" RMT

Amplitudes, probabilities in RMT

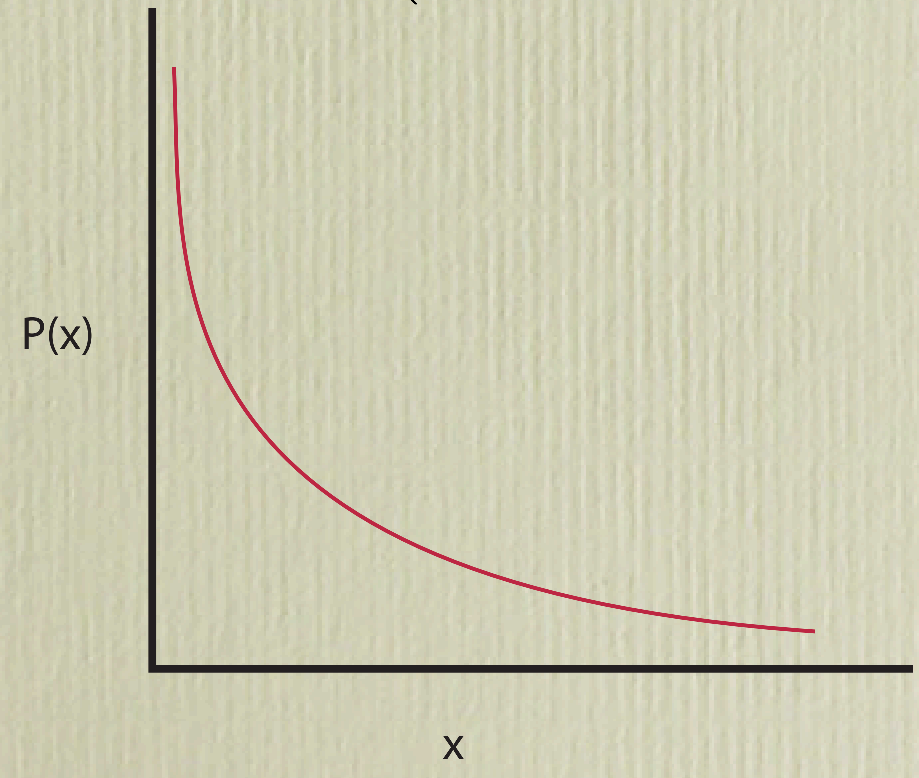
Eigenfunctions $|\psi_n\rangle$, $H|\psi_n\rangle = E_n|\psi_n\rangle$ are also Gaussian random:

$$a_n \equiv \langle\varphi|\psi_n\rangle \qquad P(a) = \sqrt{\frac{C}{2\pi}} \exp\left(-\frac{Ca^2}{2}\right)$$

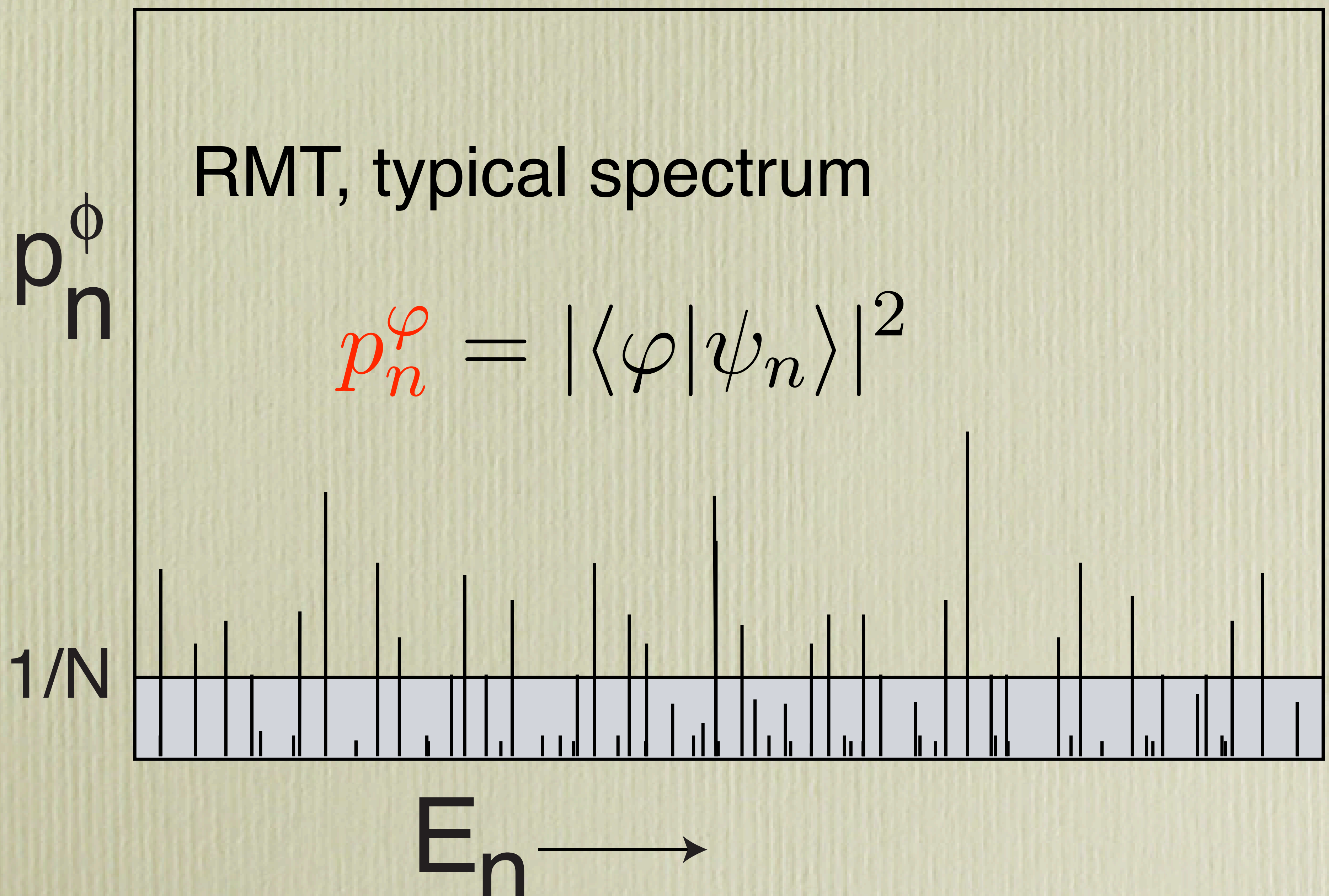
$$p_n \equiv |\langle\varphi|\psi_n\rangle|^2 \qquad P(p) = \sqrt{\frac{C}{2\pi p}} \exp\left(-\frac{Cp}{2}\right)$$

Spectra

(Porter-Thomas distribution)



RMT predicts structureless spectra, apart from random fluctuations:



Quantum Dynamics

Autocorrelation function $c(t)$

$$c(t) = \langle\varphi|e^{-iHt/\hbar}|\varphi\rangle = \langle\varphi|\varphi(t)\rangle$$

Spectrum $S(E)$

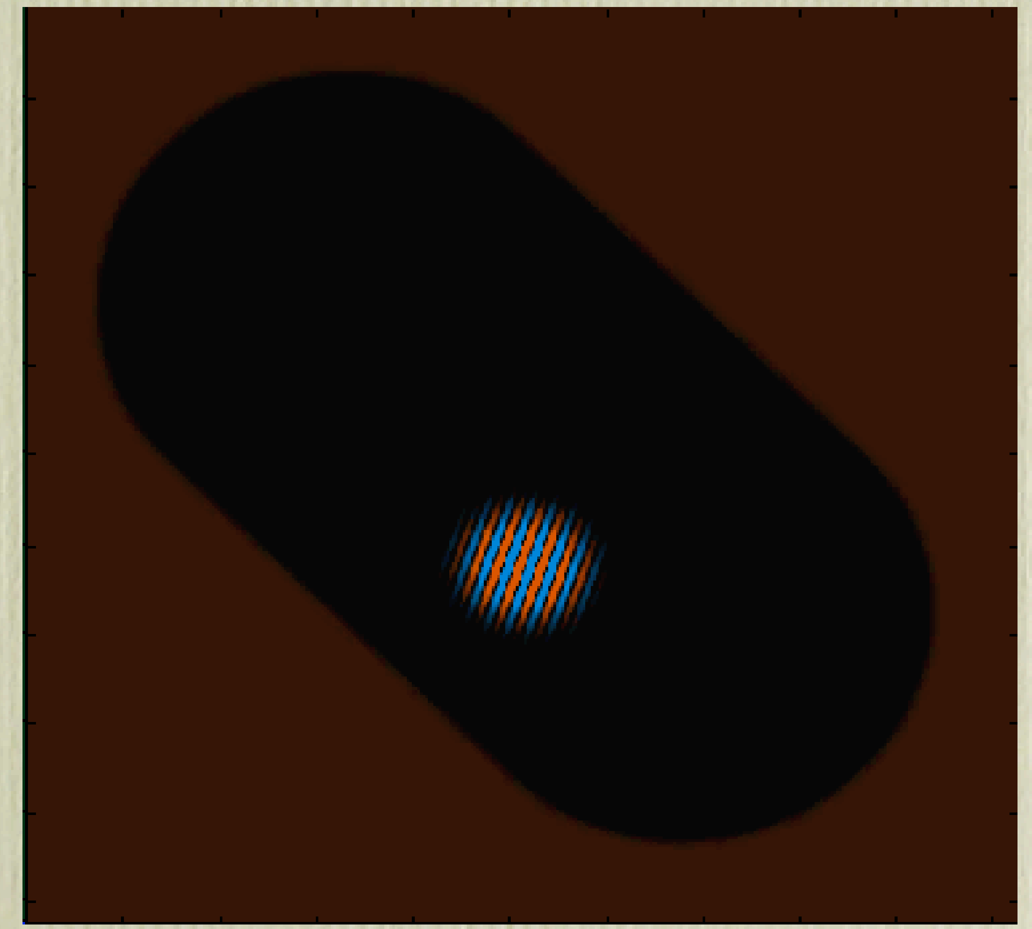
$$S(E) = \int_{-\infty}^{\infty} e^{-iEt/\hbar} c(t)\, dt = \sum_n p_n^{\varphi}\, \delta(E - E_n)$$

$$p_n^{\varphi} = |\langle\varphi|\psi_n\rangle|^2$$

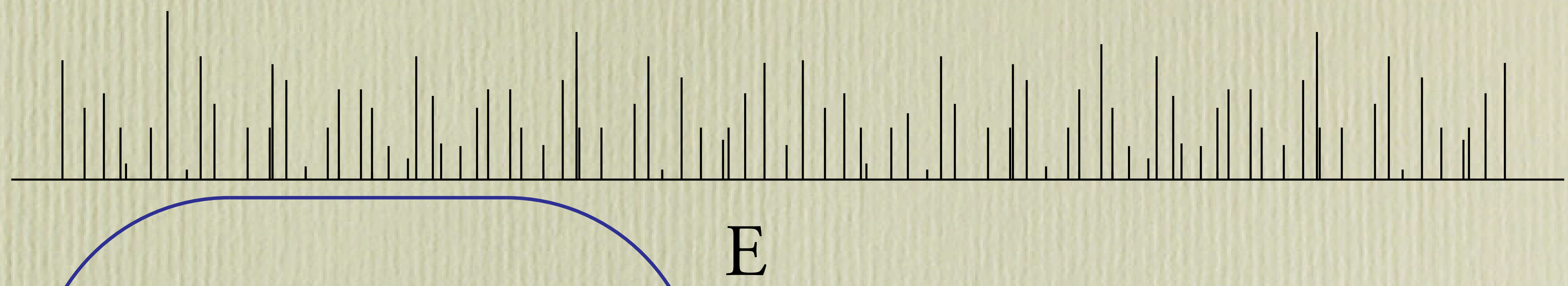
RMT "dynamics": instant decorrelation as $N \to \infty$

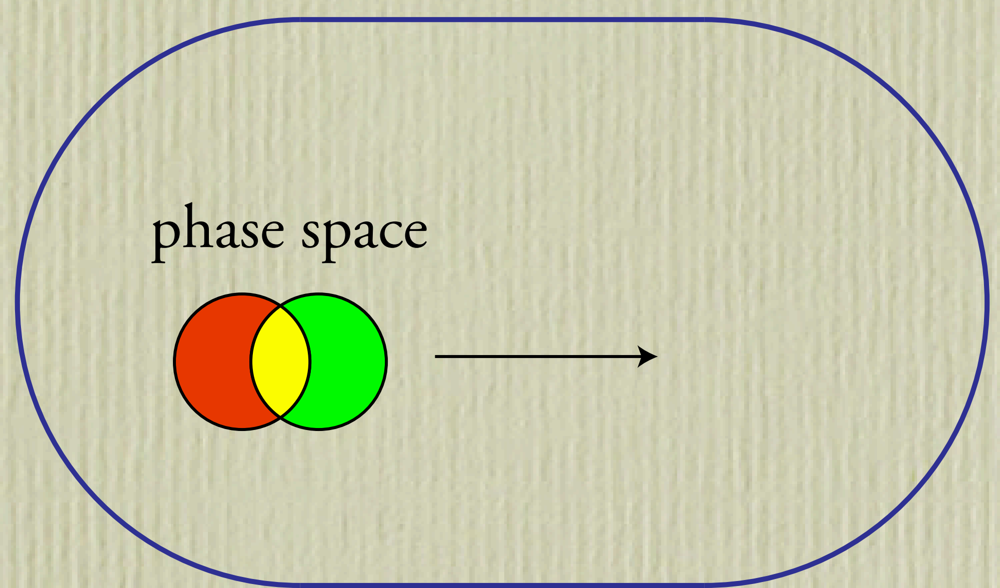
$$c(t) = 0, \quad t > 0$$

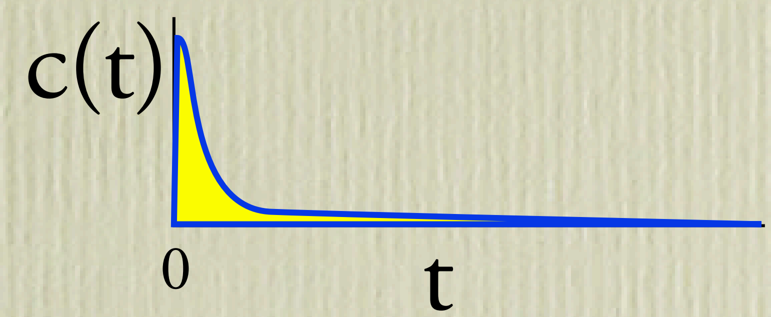
The fourier transform of the dynamics up to the Heisenberg time gives eigenfunctions to a high degree of approximation...



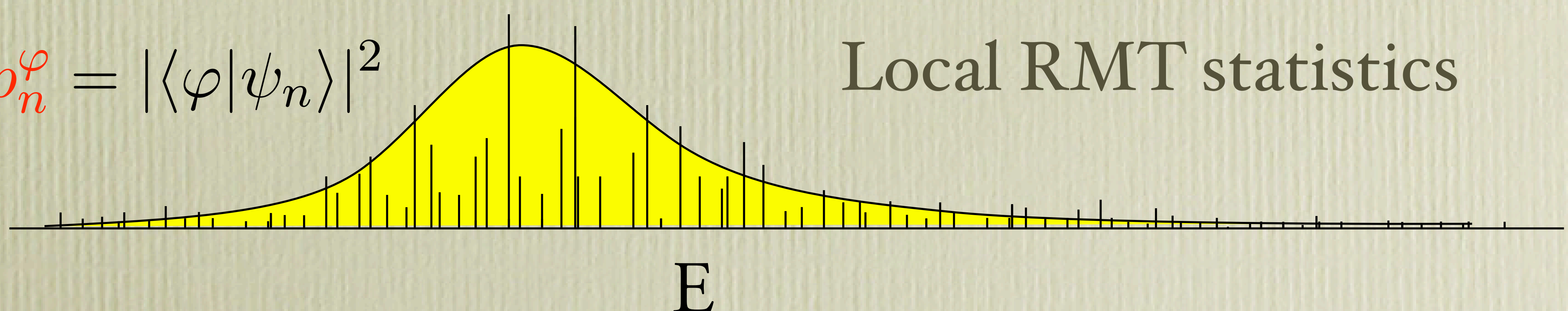
What does RMT say about the spectrum of a wavepacket?



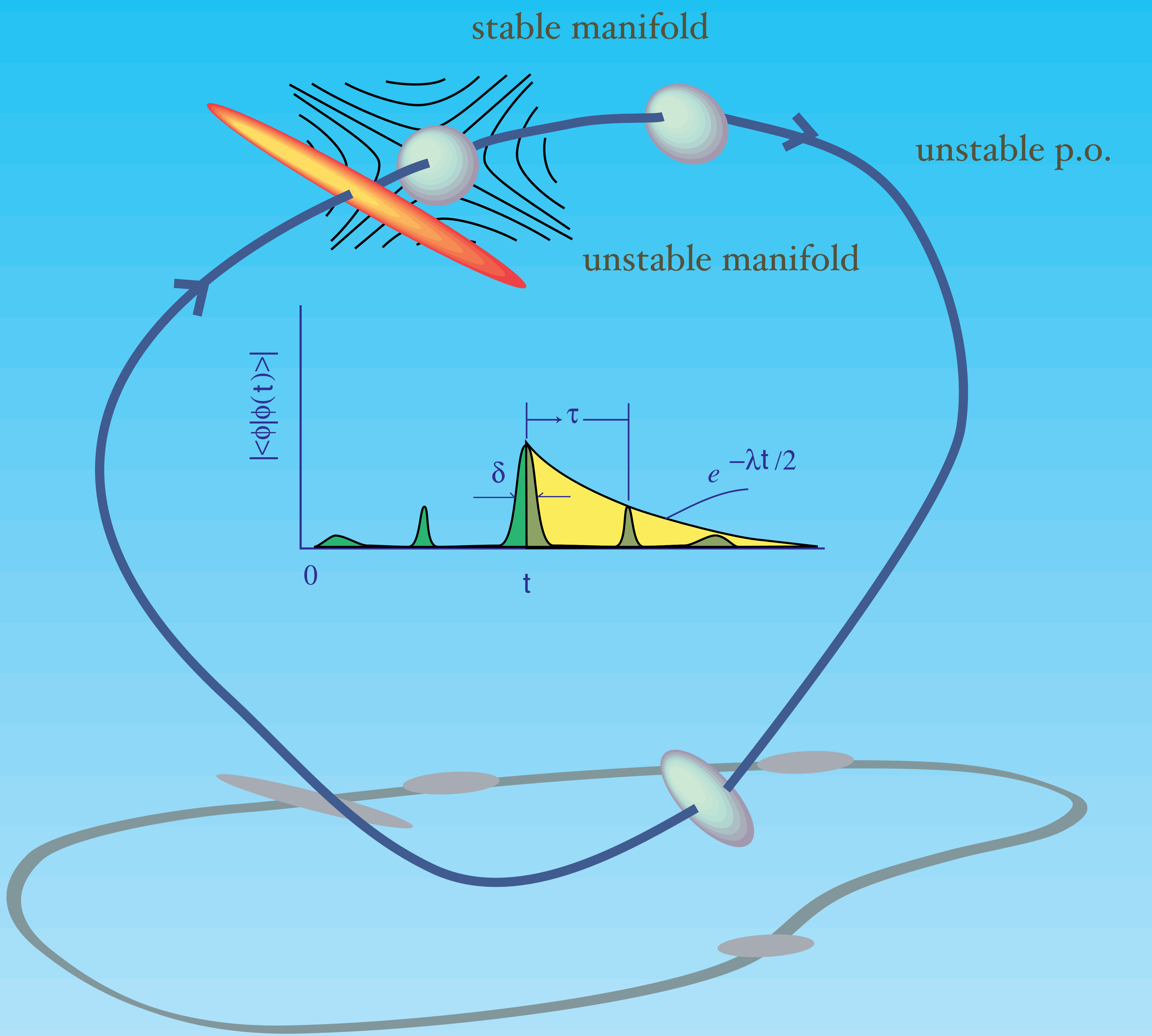




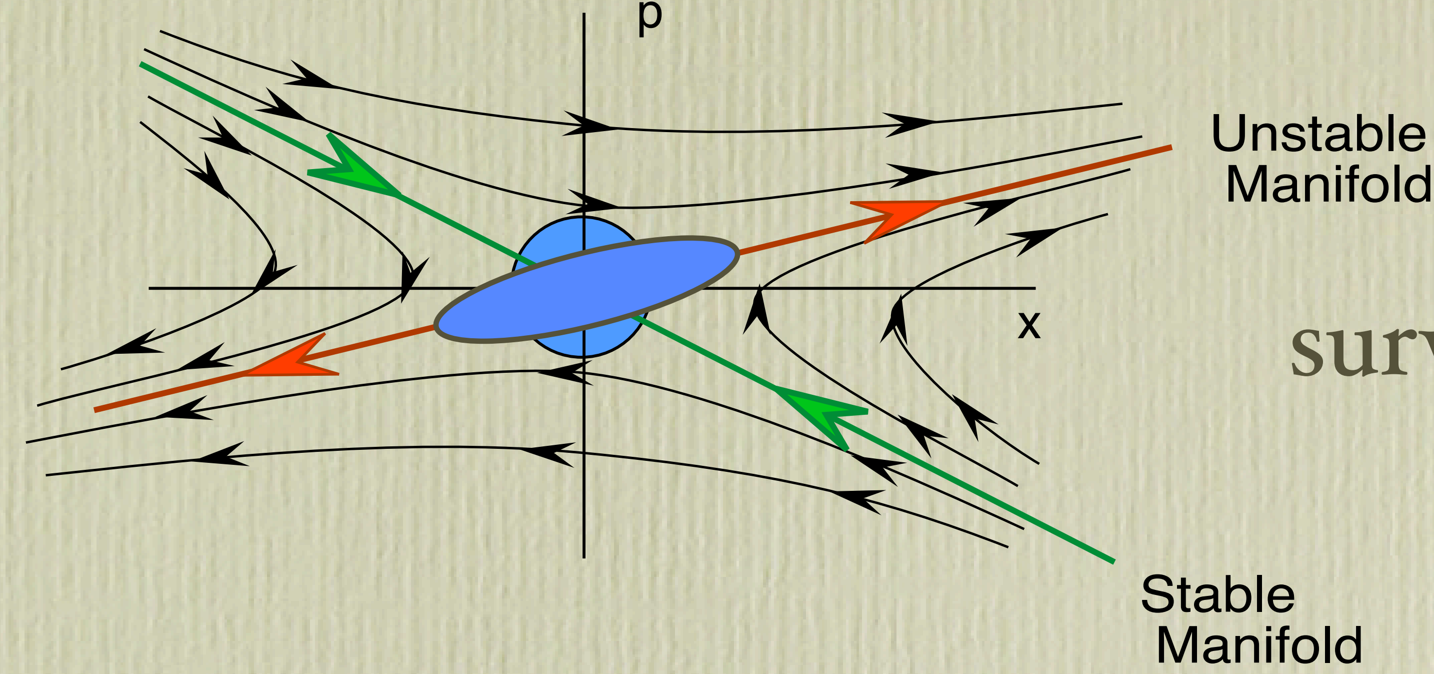
$$p_n^{\varphi} = |\langle \varphi | \psi_n \rangle|^2$$



Local RMT statistics

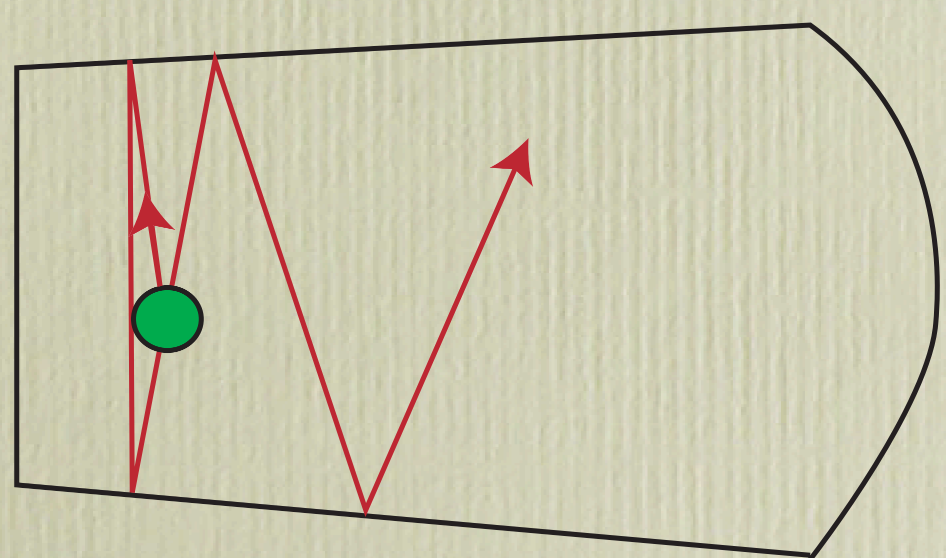

stable manifold
unstable p.o.
unstable manifold
$|\langle\phi|\phi(t)\rangle|$
δ
τ
$e^{-\lambda t/2}$
0
t

Recurrences governed by Lyopunov exponents

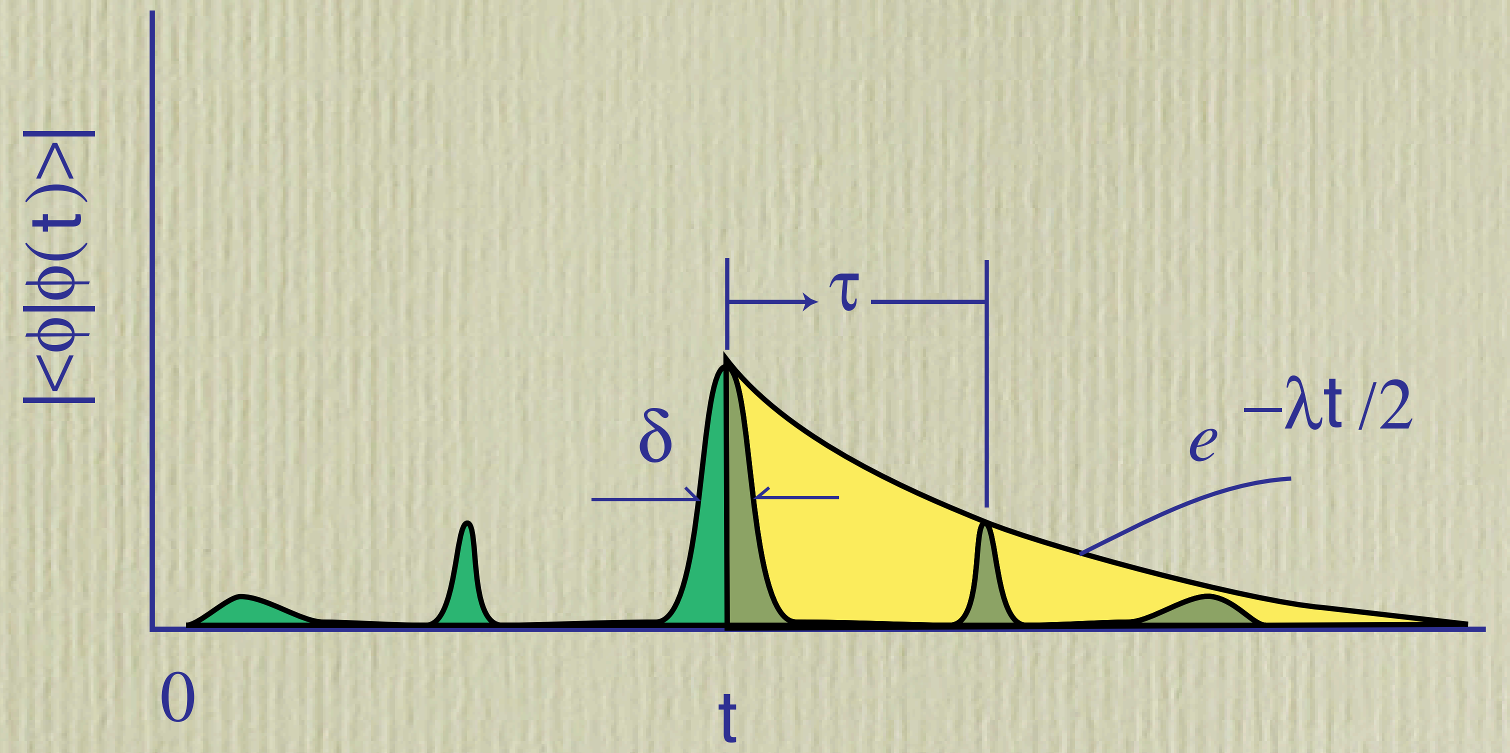


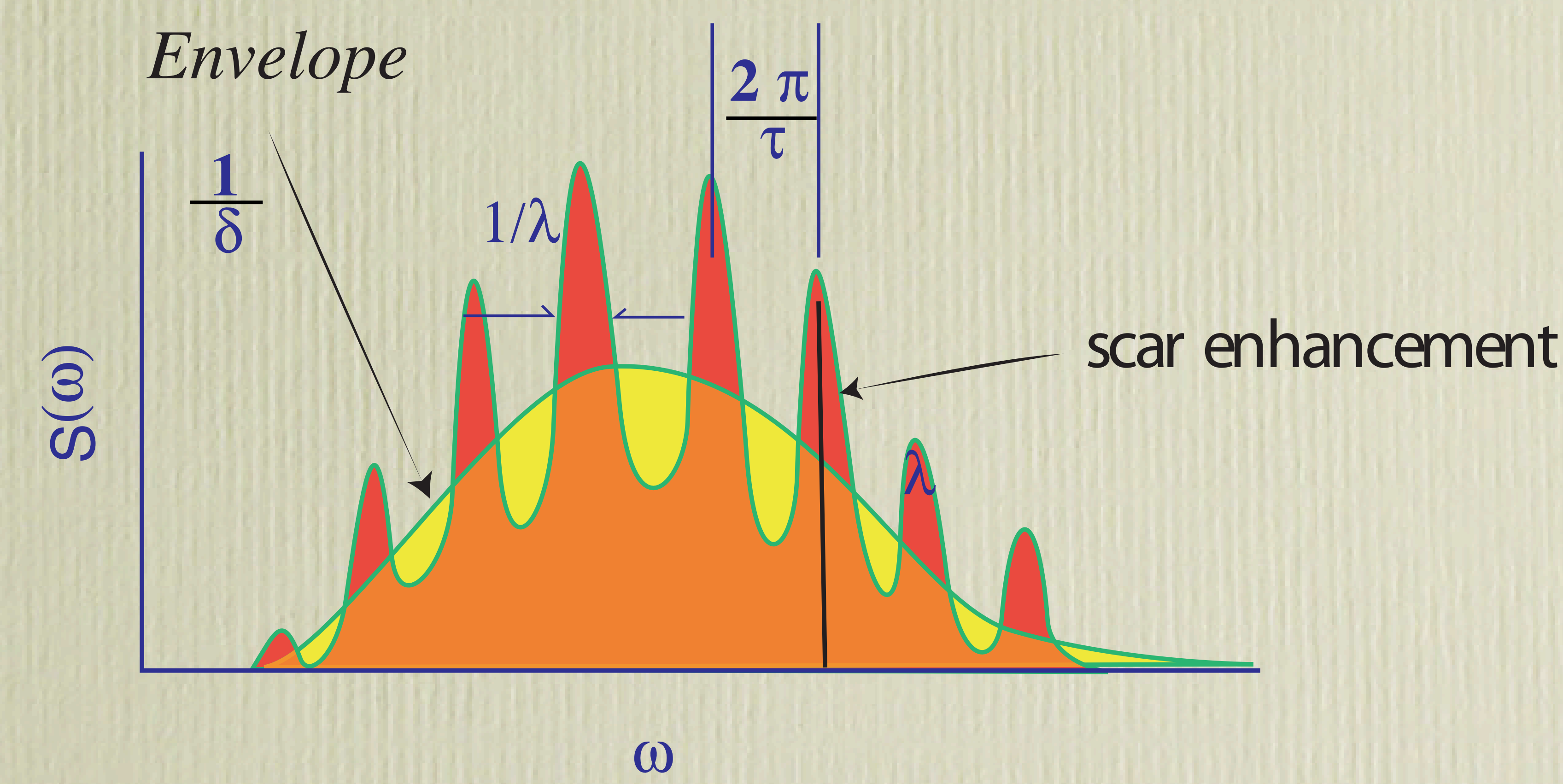
survives h → o

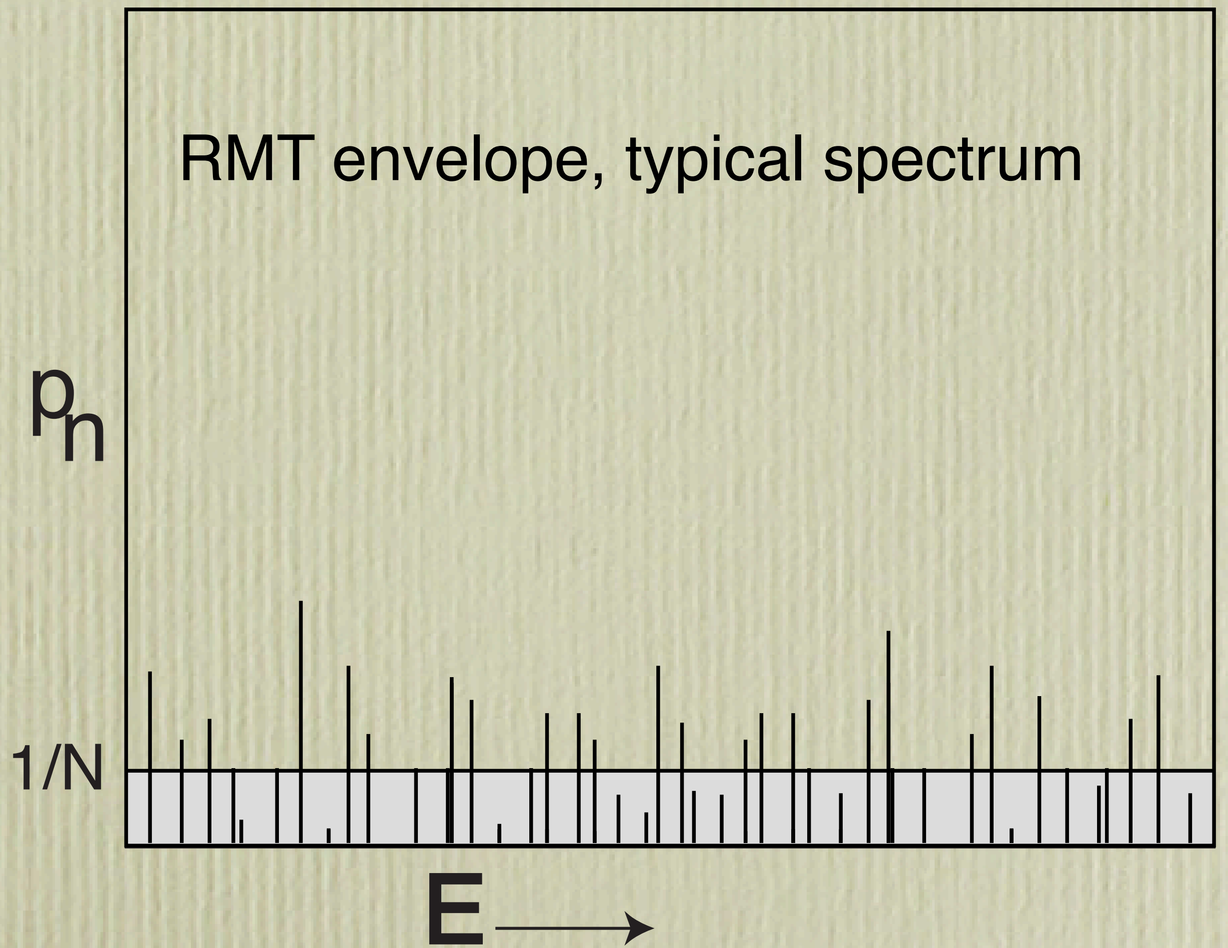
Recurrences governed by "grazing"

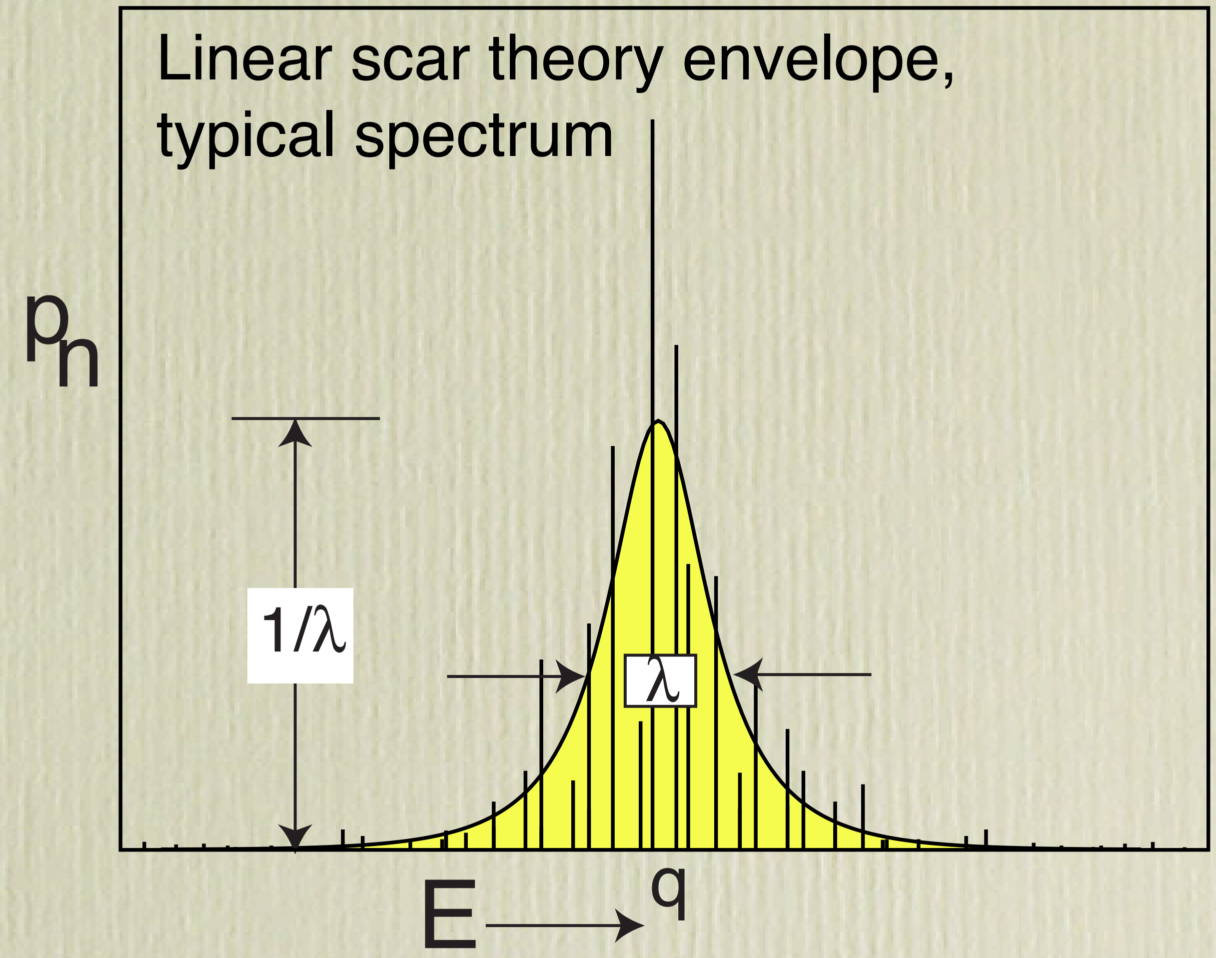


Scarring and spectra



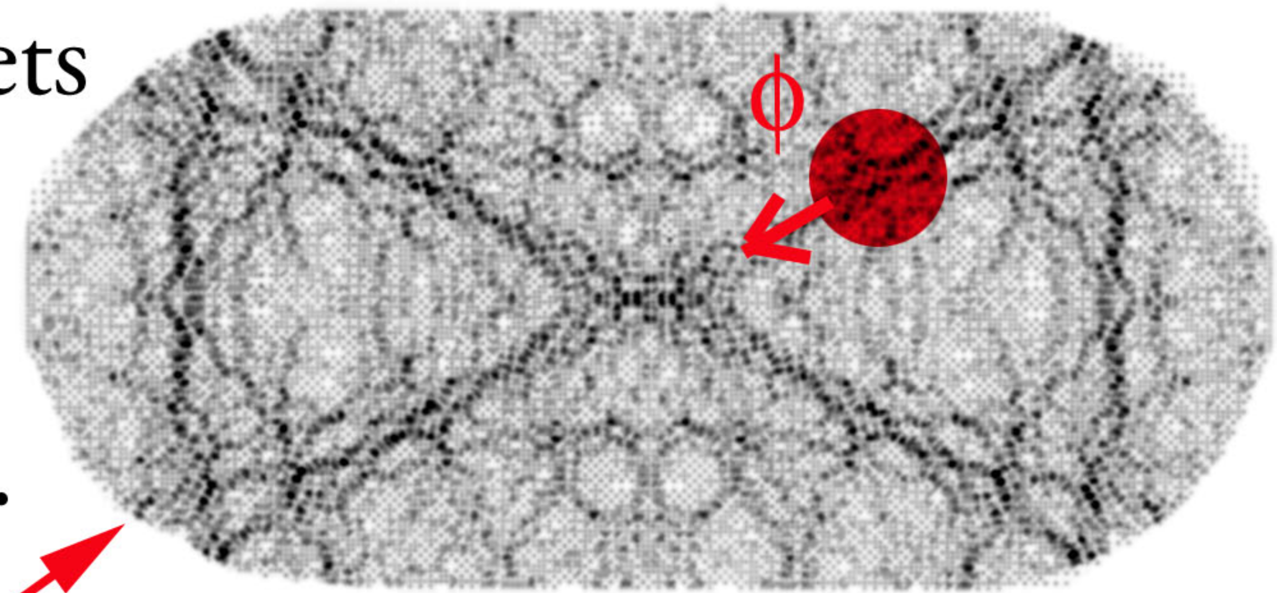



RMT envelope, typical spectrum
p_n
1/N
E


Linear scar theory envelope, typical spectrum
p_n
$1/\lambda$
λ
E
q

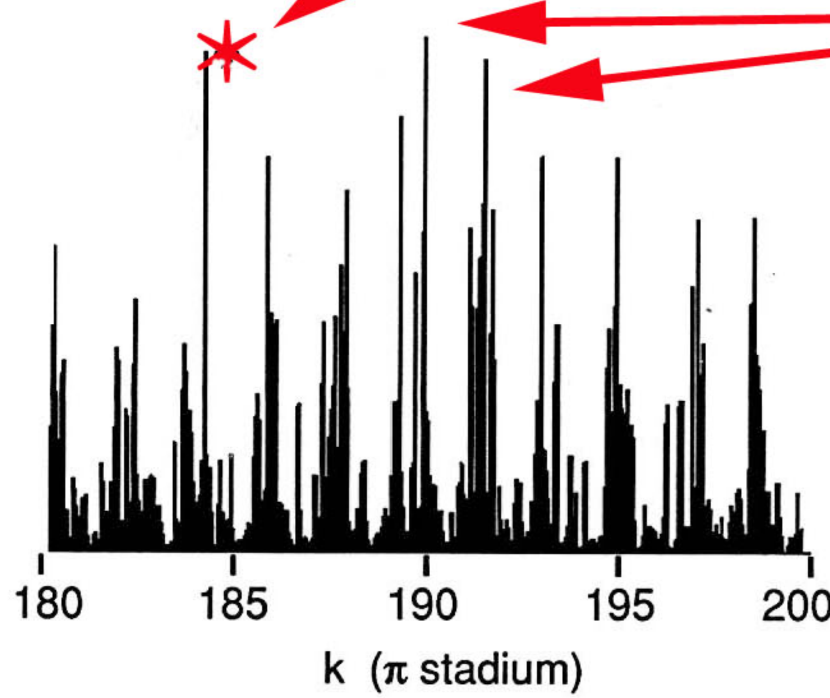
Scars and Spectra

The spectra of wavepackets launched on periodic orbits reveal scarred states as the largest peaks.



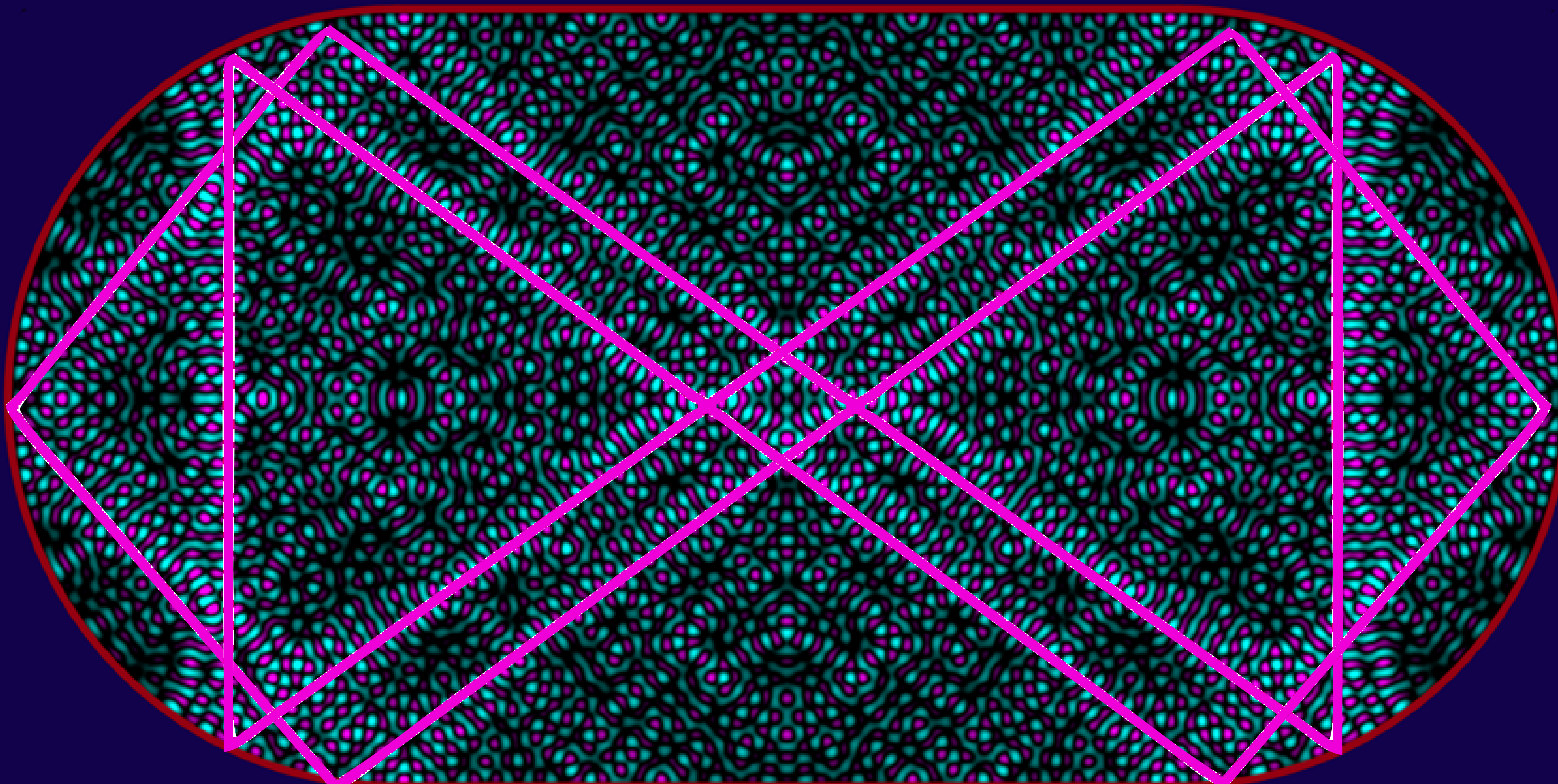
other bowtie states

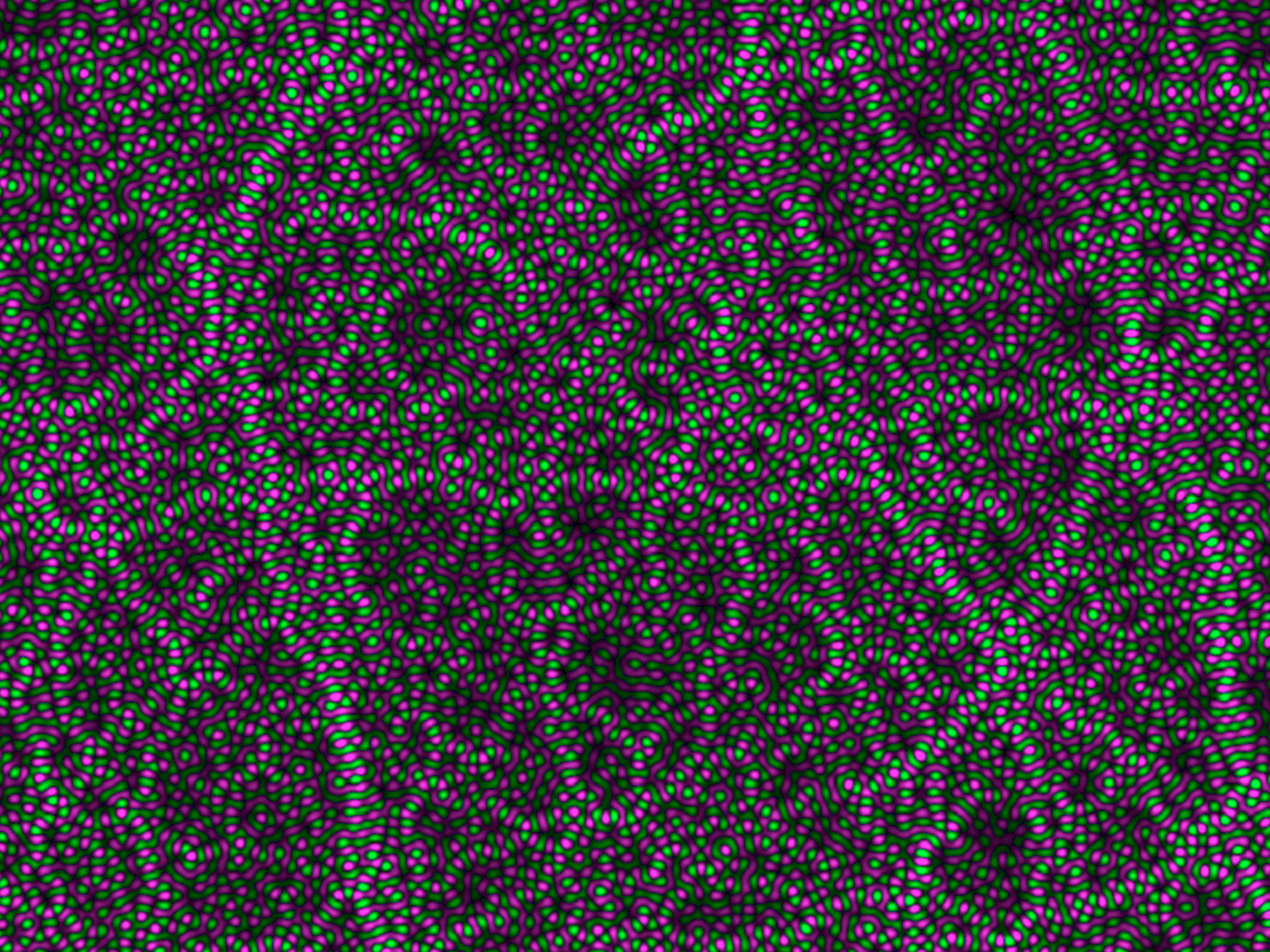
p_n^{ϕ}



$$p_n^{\phi} = |\langle n|\phi\rangle|^2$$

This is the spectrum of a wavepacket launched on the bowtie periodic orbit





Numerical test-baker's map

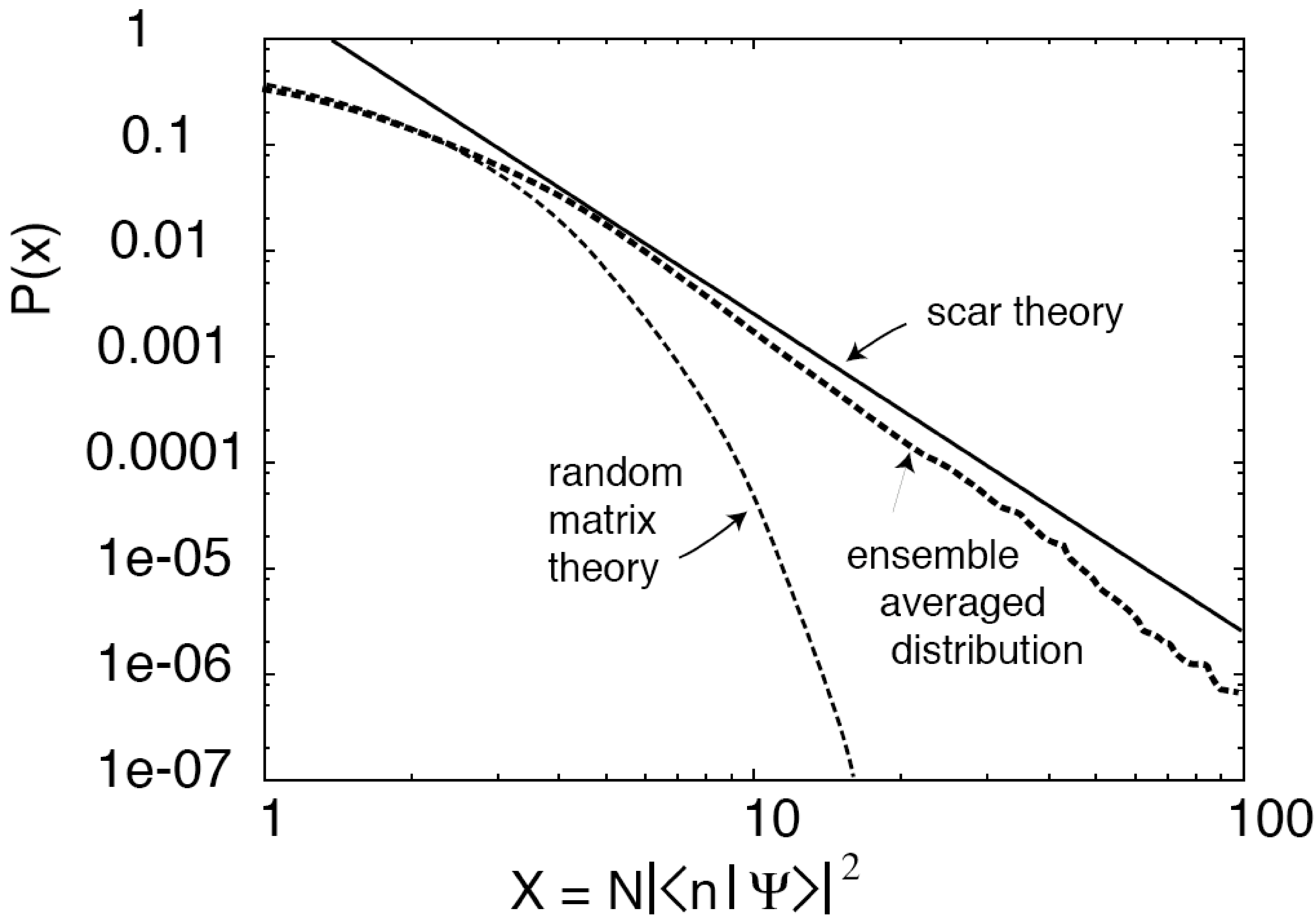
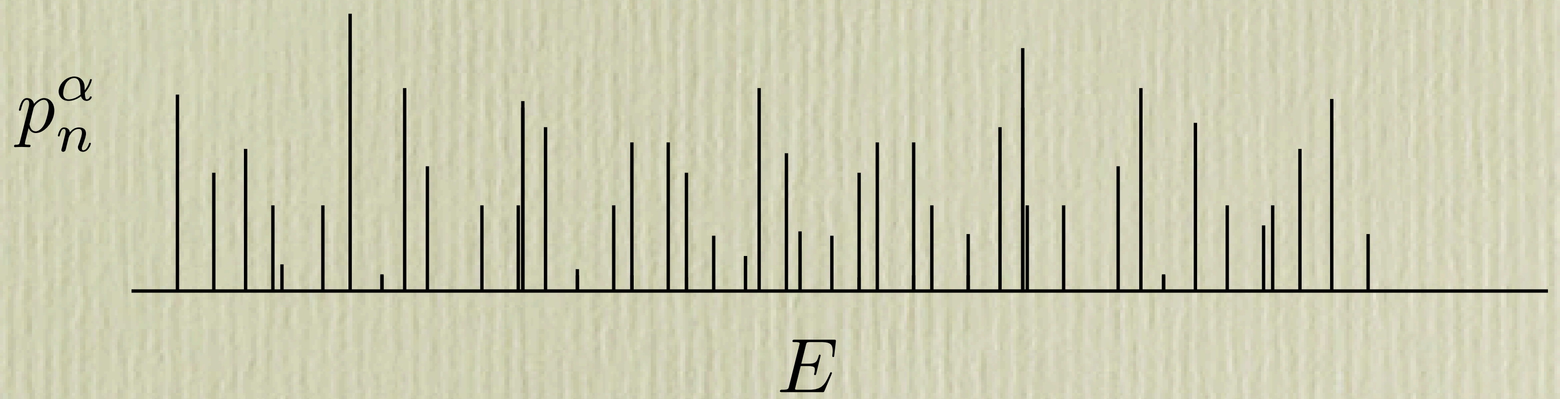


Figure 5. Cumulative wavefunction intensity distribution after ensemble averaging over systems with classical orbits of different instability exponents. The scar theory tail is given by the solid line, and the dotted curve is the RMT prediction (after [15]).

Phase Space Transport

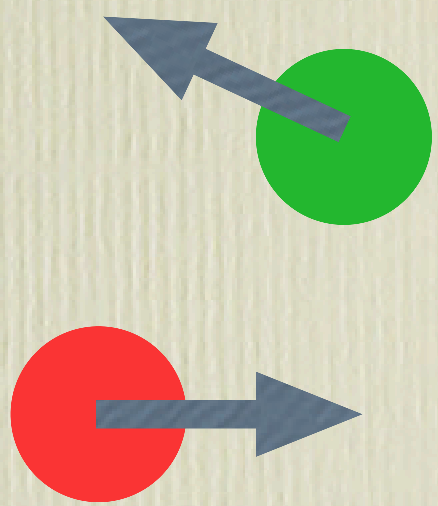
$$a_n = \langle E_n | \alpha \rangle \qquad p_n^\alpha = |\langle E_n | \alpha \rangle|^2$$

p_n^α measures the tendency of an eigenstate to be large in a certain region of phase space.



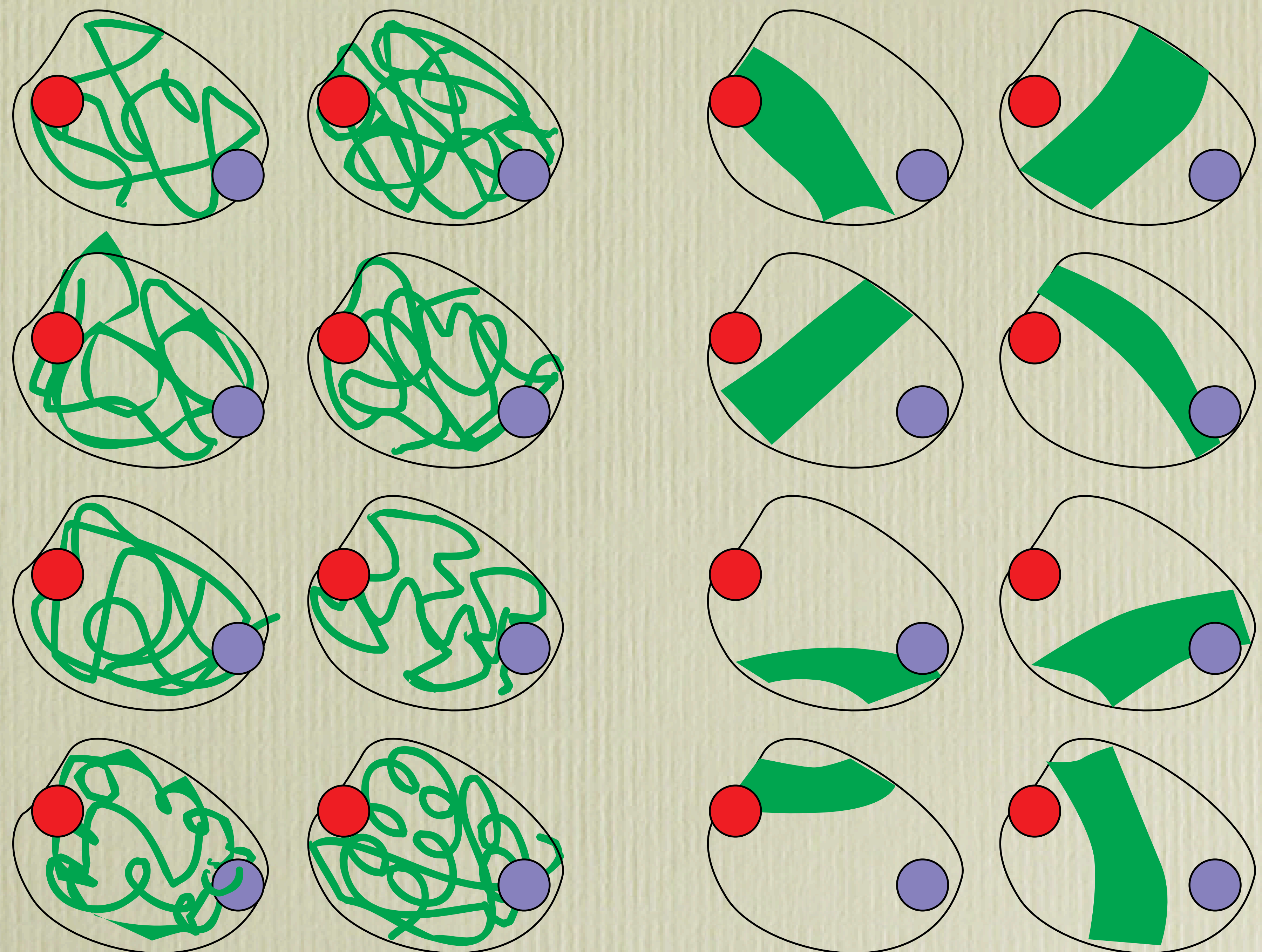
Time averaged phase space transport

$$P({\color{red}\alpha}|{\color{green}\beta}) \equiv \frac{1}{T}\int_0^{T\to\infty} |\langle {\color{green}\beta}|{\color{red}\alpha}(t)\rangle|^2 \, dt$$

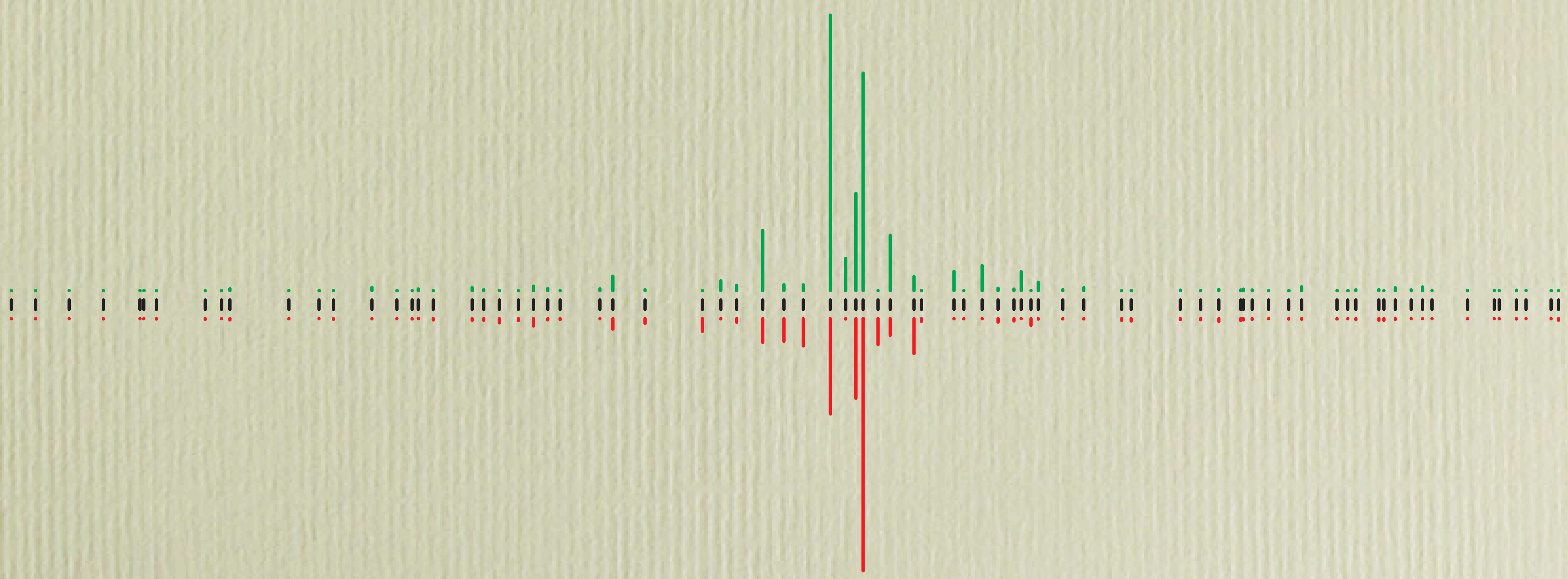


Easily shown: $P({\color{red}\alpha}|{\color{green}\beta}) = \sum p_n^{\color{red}\alpha} p_n^{\color{green}\beta}$

Chaotic $P(a|b) \sim 1/N$ Integrable $P(a|b) \sim 0$



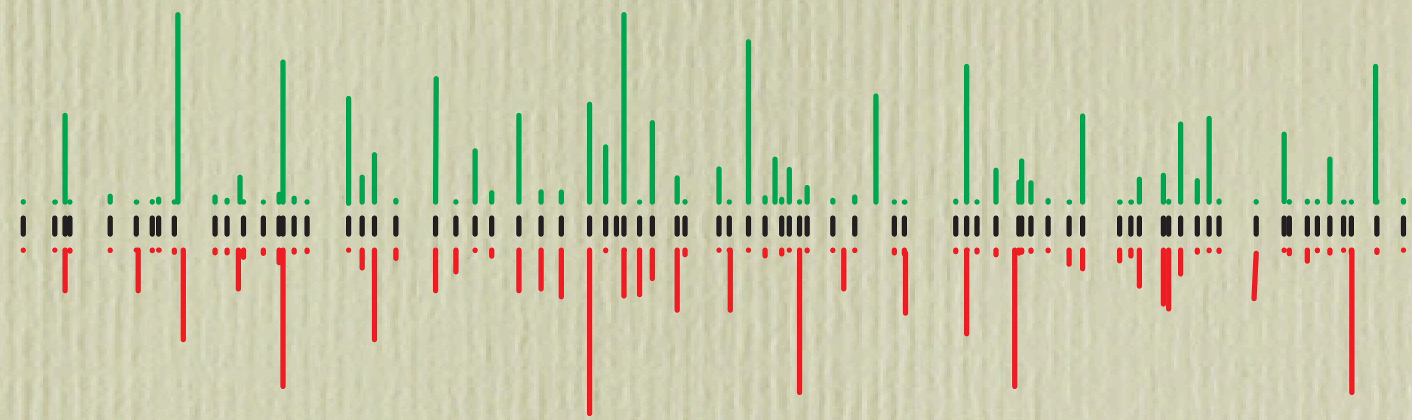
$$P(a|b) = \sum_n p_n^a p_n^b$$



$$P(a|a) = \sum_n (p_n^a)^2 = 1/\mathcal{N} = \ I.P.R.$$

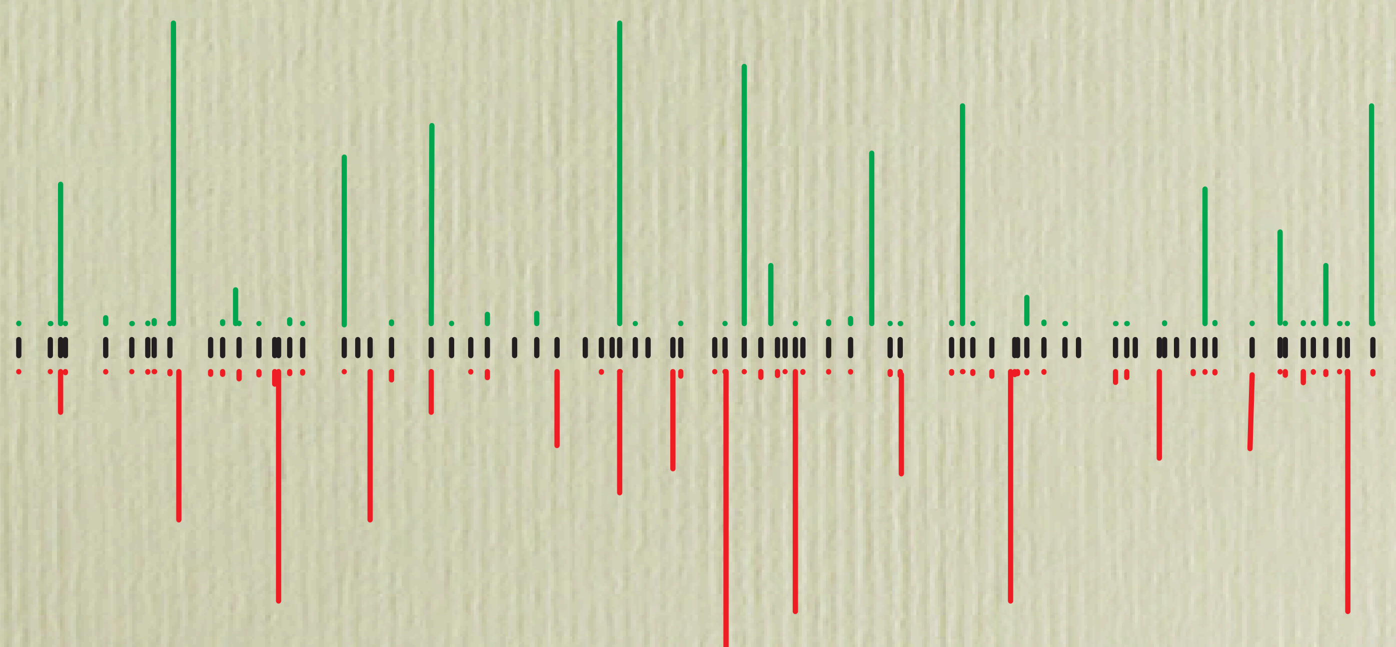
Gaussian:

RMT:



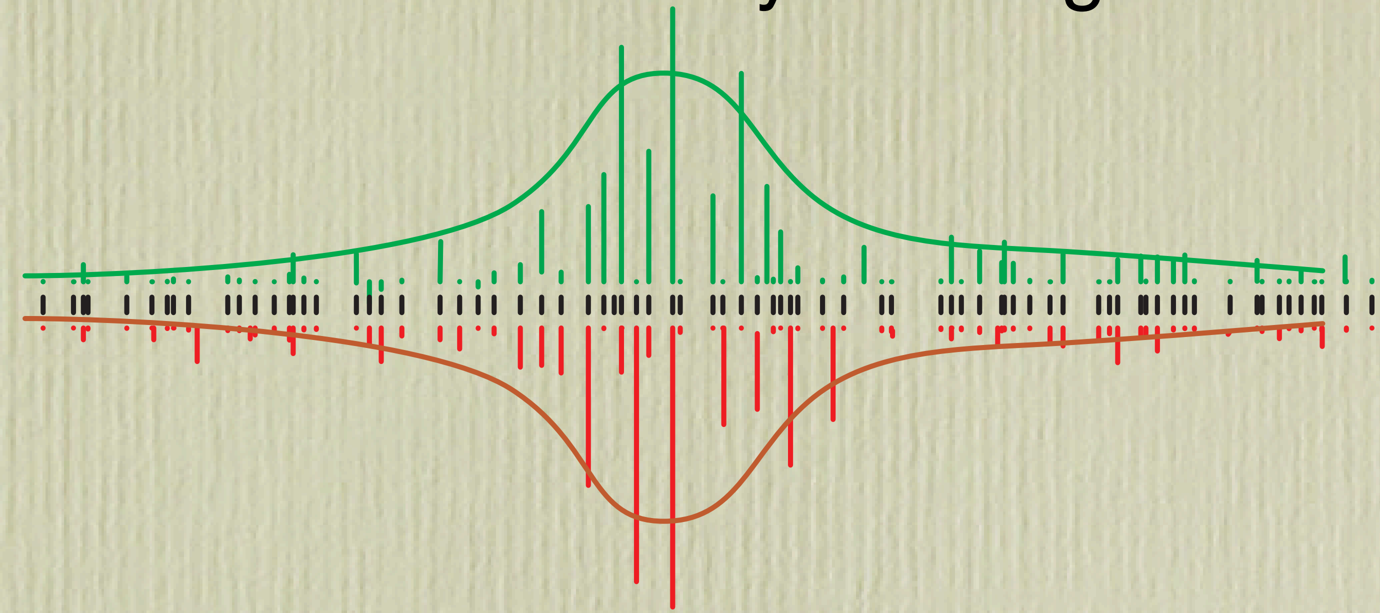
$P(a|a) = 3/N$
$P(b|b) = 3/N$
$P(a|b) = 1/N$

non-Gaussian, random:

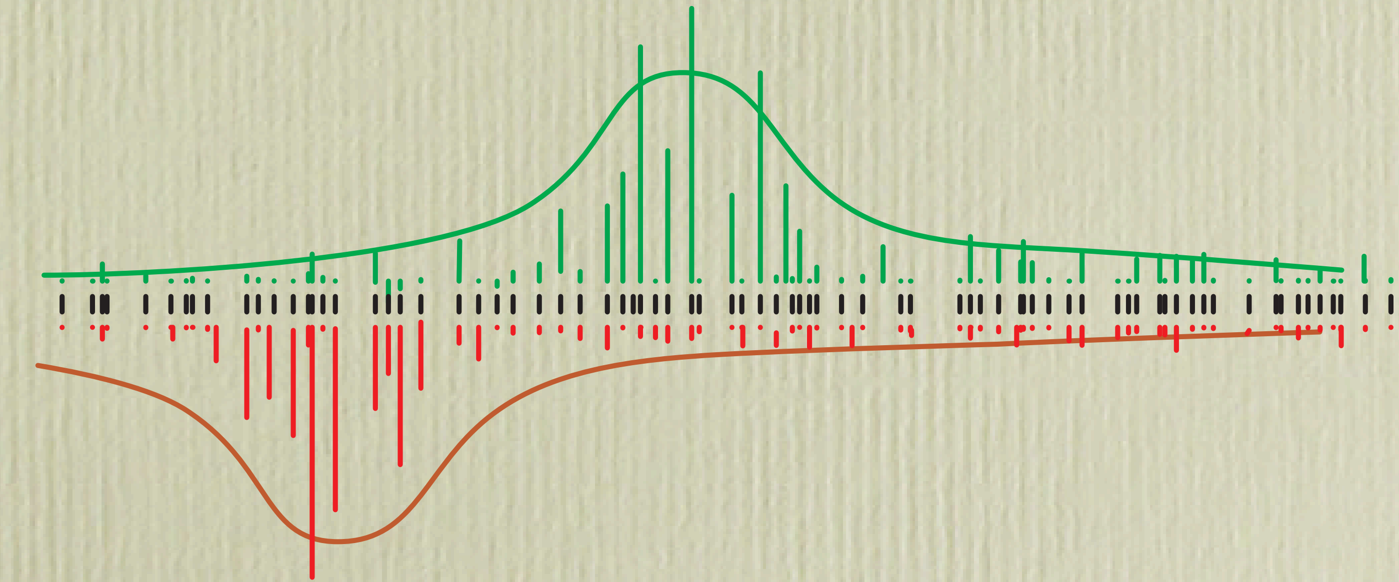


$P(a|a) = K/N; K>3$
$P(b|b) = K/N$
$P(a|b) = 1/N$

state-to-state flow affected by scarring-but only on p.o.'s



$P(a|a) = 3/(\lambda N)$
$P(b|b) = 3/(\lambda N)$
$P(a|b) = 1/(\lambda N)$

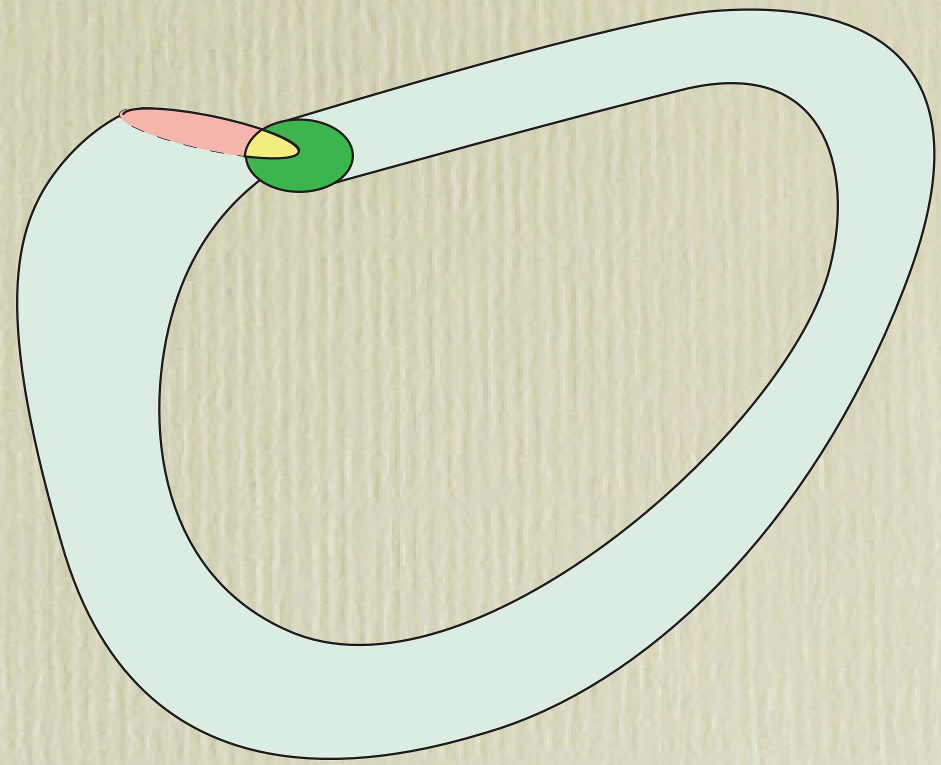


$P(a|a) = 3/(\lambda N)$
$P(b|b) = 3/(\lambda N)$
$P(a|b) << 1/(\lambda N)$

Localization implied by short time recurrences

Define volume in phase space

$$V = \frac{1}{Tr(\rho^2)}$$



Pure state density: $$\int \rho^2(\vec{p},\vec{q})\, d\vec{p}\, d\vec{q} = h^{-N},$$

Define volume accessed in phase space:

$$\rho^{av}(\vec{p},\vec{q},T) = \frac{1}{T}\int_0^T \rho(\vec{p},\vec{q},t)dt$$

This average density will obey

$$h^N Tr[(\rho^{av})^2] \equiv \frac{1}{\mathcal{N}_T} \leq 1$$

This defines $\mathcal{N}_T$, the number of phase space cells accessed.

We have

$$\frac{1}{\mathcal{N}_T} = h^N Tr[(\rho^{av})^2] = \frac{2}{T}\int_0^T (1-\frac{\tau}{T})P(\tau)\,d\tau$$

where the survival probability $P(\tau)$ is

$$P(\tau) = h^N Tr[\rho(0)\rho(\tau)]$$

Note that in the case that ρ corresponds to a pure quantum state density $|\varphi\rangle\langle\varphi|$, P is obtainable from the spectral distribution $S(E)$ of the state $|\varphi\rangle$, as follows:

$$S(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{iEt/\hbar}\langle\varphi|\varphi(t)\rangle\,dt$$

By inverse Fourier transform,

$$\langle\varphi|\varphi(t)\rangle = \int_{-\infty}^{\infty} e^{-iEt/\hbar} S(E)\,dE,$$

and of course $P(t) = |\langle\varphi|\varphi(t)\rangle|^2$

The rate $\mathcal{R}$ is easily shown, by differentiation, to be

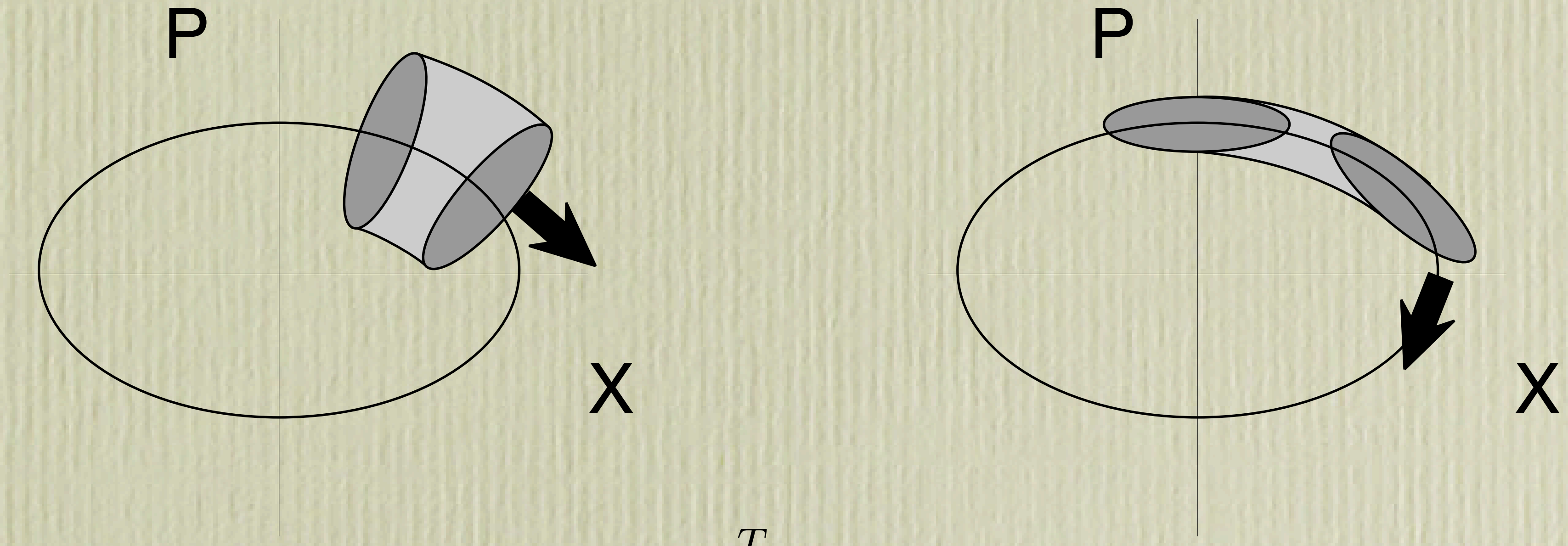
$$\mathcal{R} = \frac{\frac{1}{2}\int_0^T (1 - \frac{2\tau}{T}) P(\tau) d\tau}{\left(\int_0^T (1 - \frac{\tau}{T}) P(\tau)\, d\tau\right)^2}$$

steady state rate (for large enough T and between any recurrences)

$$\mathcal{R} = \frac{1}{2\int_0^T P(\tau)\, d\tau} = \frac{1}{\int_{-T}^{T} P(\tau)\, d\tau}$$

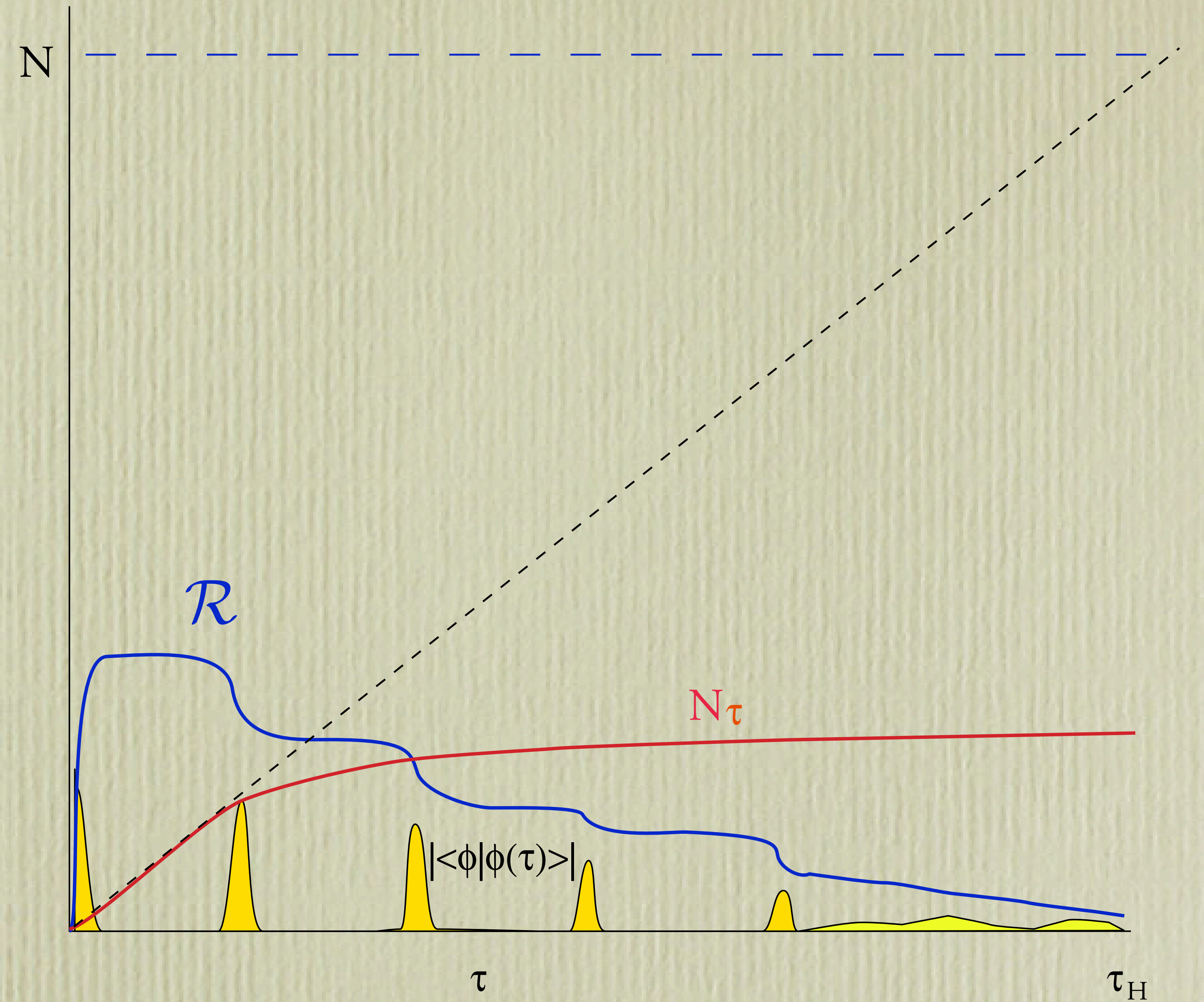
$$\text{Volume occupied} = \frac{1}{Tr(\rho^2)}$$

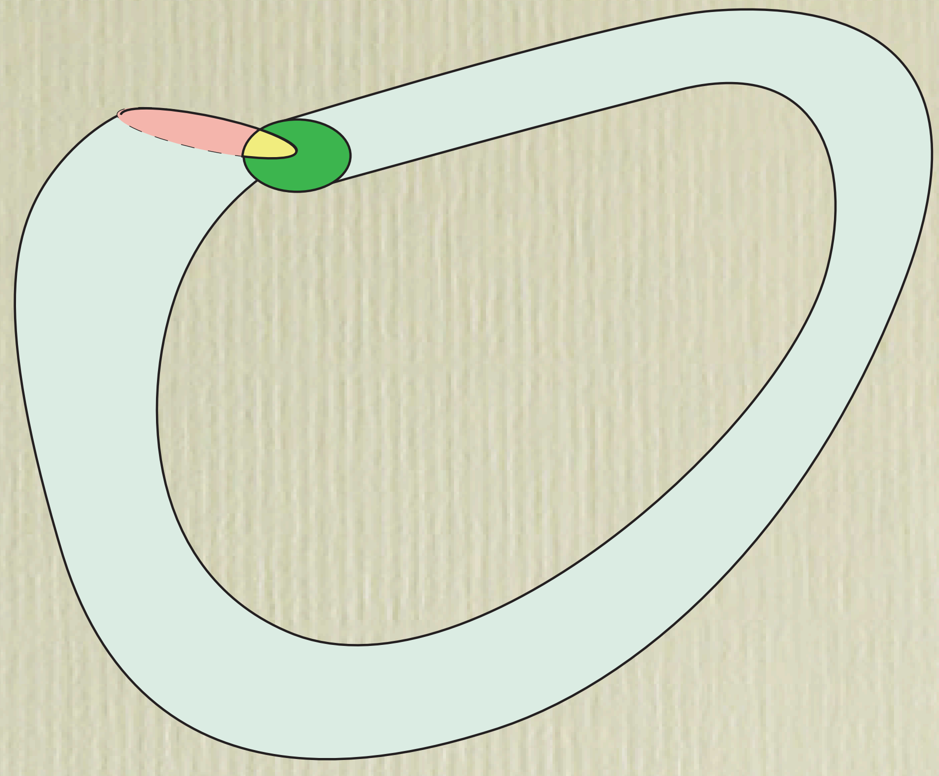
$$\rho^{av}(\vec{p},\vec{q},T) = \frac{1}{T}\int_0^T \rho(\vec{p},\vec{q},t)dt$$



$$\frac{1}{\mathcal{N}_T} = h^N Tr[(\rho^{av})^2] = \frac{2}{T}\int_0^T (1-\frac{\tau}{T})P(\tau)\,d\tau$$

$$P(\tau) = h^N Tr[\rho(0)\rho(\tau)]$$





Strong recurrences slow down the exploration of new phase space. They happen classically too, also slowing exploration, but there is no time limit. The quantum system must explore before the Heisenberg time

Implications

- Scarring and related phenomena affect conductance fluctuations, wavefunction statistics, ...
- RMT needs modification and is not a “sufficient” theory for classically chaotic systems.
- However, RMT is a great starting point and the mantra is “the system is random subject to the *a priori* known constraints”
- There is no proof the eigenfunctions of classically chaotic systems are Gaussian random; some special slowly mixing cases are known where they are not.

Quantum chaos is now a large and established field. Other topic include

- connections of QC to Riemann Zeta function
- semiclassical asymptotics supporting RMT
- nonlinear sigma model, supersymmetry (powerful approach to RMT results)
- experiments
- ...

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